13. Sheaves of modules

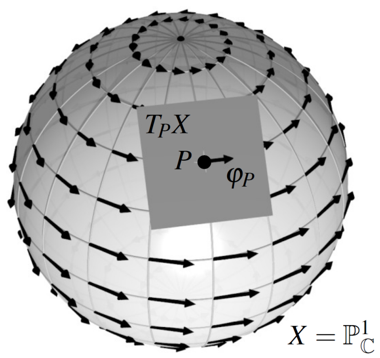
Big Picture

- R ring $\rightsquigarrow$ $\operatorname{Spec} R$ affine scheme $\rightsquigarrow$ X scheme $=$ locally $\operatorname{Spec} R$
  ↳ geometric interpretation for theory of rings:
  - surjection $S \twoheadrightarrow R$ $\rightsquigarrow$ closed embedding $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} S$
  - localization $R \to R_f$ $\rightsquigarrow$ distinguished open $\operatorname{Spec} R_f \subseteq \operatorname{Spec} R$

  ↳ could continue in this direction:
  $S \subseteq R$ finite / integral / flat / ... extension of rings $\rightsquigarrow$ prop. of $\operatorname{Spec} R \to \operatorname{Spec} S$
  $\rightsquigarrow$ prop. of $X \to Y$
  schemes
- Alternative direction: What about theory of modules?
  ↳ M an R-module $\rightsquigarrow$ gives structure of sheaf $\tilde{M}$ on $\operatorname{Spec} R$.
  (e.g. $M = R \rightsquigarrow \tilde{M} = \mathcal{O}_{\operatorname{Spec} R}$ structure sheaf)
- <u>Goals</u> Learn more about sheaves (of modules) & generalize constructions from comm. algebra ($\ker$, im, $\otimes$, ...).

Why care about sheaves?

<u>Exa</u> (Tangent sheaf)
X smooth curve (e.g. $X = \mathbb{P}^1_{\mathbb{C}}$ Riemann sphere)
$p \in X \rightsquigarrow T_pX$ tangent space $(\cong K^1)$



$$\rightsquigarrow \underbrace{T_X(U)}_{U \subseteq X \text{ open}} = \left\{ \varphi = (\varphi_p)_{p \in U} : \begin{array}{l} \varphi_p \in T_pU \text{ "varying} \\ \text{nicely with } p\text{"} \end{array} \right\}$$

tangent sheaf

φ: tangent vector field

<u>Extra structure</u> $\psi \in \mathcal{O}_X(U) \rightsquigarrow \psi \cdot \varphi = (\underbrace{\psi(p)}_{\in K} \cdot \underbrace{\varphi_p}_{\in T_pU})_{p \in U} \in T_X(U)$
$\rightsquigarrow T_X(U)$ is $\mathcal{O}_X(U)$-module

Def ((Pre-) sheaves of modules)
X scheme.
(a) A <u>(pre-) sheaf of modules</u> on X
is a (pre-) sheaf $\mathcal{F}$ on X s.t. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module $\forall U \subseteq X$ open,
such that all restrict. maps are $\mathcal{O}_X$-module homomorphisms:

$$(\varphi+\psi)|_V = \varphi|_V + \psi|_V \quad \text{and} \quad (\lambda\cdot\varphi)|_V = \lambda|_V \cdot \varphi|_V \quad \text{for } V \subseteq U \subseteq X \text{ opens}$$
$\varphi, \psi \in \mathcal{F}(U), \lambda \in \mathcal{O}_X(U)$

<u>Notation</u> <u>$\mathcal{O}_X$-module</u> = sheaf of modules, sometimes: omit "of modules"

(b) $f: \mathcal{F} \to \mathcal{G}$ morphism of (pre-) sheaves of modules on X
given by:
$$f_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \quad \mathcal{O}_X(U)\text{-module homom. } \forall U \subseteq X \text{ open}$$
compatible with restrictions:

$$f_U(\varphi)|_V = f_V(\varphi|_V) \qquad \forall V \subseteq U \subseteq X \text{ open}, \ \varphi \in \mathcal{F}(U).$$

<u>Notation</u> Write f_U as f where no risk of confusion.

<u>Exa</u> (a) $\mathcal{F} = \mathcal{O}_X$ is sheaf of modules on X.
(b) $\mathcal{F}, \mathcal{G}$ two $\mathcal{O}_X$-modules $\rightsquigarrow$ $\mathcal{F} \oplus \mathcal{G}$ <u>direct sum</u> $\mathcal{O}_X$-module:

$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$
direct sum as $\mathcal{O}(U)$-modules

Twisting sheaves on projective space

Let's see our first non-trivial example of a sheaf of $\mathcal{O}_X$-modules on $X = \mathbb{P}^n$.

Construction (Twisting sheaves on $\mathbb{P}^n$)

$n \in \mathbb{N}$, $d \in \mathbb{Z}$. For $\emptyset \neq U \subseteq \mathbb{P}^n$ we define

$$(\mathcal{O}_{\mathbb{P}^n}(d))(U) := \left\{ \frac{g}{f} : \begin{array}{l} f \in K[x_0,\dots,x_n]_e \text{ and } g \in K[x_0,\dots,x_n]_{e+d} \text{ for some } e \in \mathbb{Z} \\ \text{such that } f(p) \neq 0 \ \forall p \in U \end{array} \right\}$$

$\subseteq \operatorname{Frac}(K[x_0,\dots,x_n])$. set $\mathcal{O}_X(\emptyset) = \{0\}$.

Then we have:

(a) $\mathcal{O}_{\mathbb{P}^n}(d)$ is a sheaf (of sets):

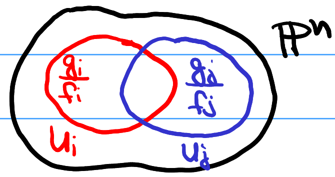
all $\mathcal{O}_{\mathbb{P}^n}(d)(U)$ are subsets of $\operatorname{Frac}(K[x_0,\dots,x_n])$, restriction = identity:

$$\mathcal{O}_{\mathbb{P}^n}(d)(U) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d)(V), \quad \frac{g}{f} \mapsto \frac{g}{f} \quad \text{for } V \subseteq U \subseteq \mathbb{P}^n \text{ open.}$$

$f(p) \neq 0 \ \forall p \in U \Rightarrow f(p) \neq 0 \ \forall p \in V$

↳ restriction compatible with composition $(\varphi|_V|_W = \varphi|_W)$ ⇝ presheaf.

↳ $\{U_i : i \in I\}$ open cover of $U \subseteq \mathbb{P}^n$ with $\frac{g_i}{f_i} \in \mathcal{O}_{\mathbb{P}^n}(d)(U_i)$ compatible on overlaps $U_i \cap U_j$



$\Longrightarrow$ For $U_i \neq \emptyset, U_j \neq \emptyset$: $U_i \cap U_j \neq \emptyset$ ($\mathbb{P}^n$ irreducible)

so $\frac{g_i}{f_i} = \frac{g_j}{f_j} \in \operatorname{Frac}(K[x_0,\dots,x_n])$

$\Rightarrow \exists$ unique section $\frac{g}{f} \in \mathcal{O}_{\mathbb{P}^n}(d)(U)$ restrict. to $\frac{g_i}{f_i}$ on U_i ⇝ sheaf.

(b) $\mathcal{O}_{\mathbb{P}^n}(d)$ is a sheaf of abelian groups (but not rings!)

$$\underset{\deg e_1}{\frac{\overset{\deg e_1 + d}{g_1}}{f_1}} + \underset{\deg e_2}{\frac{\overset{\deg e_2 + d}{g_2}}{f_2}} = \underbrace{\frac{g_1 f_2 + g_2 f_1}{f_1 \cdot f_2}}_{\text{sect. of } \mathcal{O}_{\mathbb{P}^n}(d)} \begin{array}{l} \leftarrow \deg e_1+e_2+d \\ \leftarrow \deg e_1+e_2 \end{array}$$

but $\underbrace{\frac{g_1 g_2}{f_1 \cdot f_2}}_{\text{sect. of } \mathcal{O}_{\mathbb{P}^n}(2d)!} \begin{array}{l} \leftarrow \deg e_1+e_2+2d \\ \leftarrow \deg e_1+e_2 \end{array}$

(c) For $d=0$ we have: $\mathcal{O}_{\mathbb{P}^n}(0) \cong \mathcal{O}_{\mathbb{P}^n}$

$$(\mathcal{O}_{\mathbb{P}^n}(0))(U) := \left\{ \frac{g}{f} : \begin{array}{l} f \in K[x_0,\dots,x_n]_e \text{ and } g \in K[x_0,\dots,x_n]_e \quad \text{for some } e \in \mathbb{Z} \\ \text{such that } f(p) \neq 0 \ \forall p \in U \end{array} \right\}$$

$$\begin{array}{c} (\mathcal{O}_{\mathbb{P}^n}(0))(U) \\ \Phi \Big\downarrow \begin{array}{l} g/f \\ \ \ \rotatebox{90}{$\mapsto$} \\ g/f \end{array} \\ \mathcal{O}_{\mathbb{P}^n}(U) \end{array}$$

$\rightsquigarrow$ Φ well-defined ($\deg f = \deg g$), injective ($\frac{g}{f} = 0$ on $U \subseteq \mathbb{P}^n$ dense open $\Rightarrow g = 0$)

Φ surjective: $\varphi \in \mathcal{O}_{\mathbb{P}^n}(U) \rightsquigarrow \exists$ open cover $\{U_i : i \in I\}$ of U with $\varphi|_{U_i} = \frac{g_i}{f_i}$ $(*)$ (wlog: $U_i \neq \emptyset$)

Equality $\frac{g_i}{f_i}\Big|_{U_i \cap U_j} = \frac{g_j}{f_j}\Big|_{U_i \cap U_j}$

$\rightsquigarrow g_i f_j - g_j f_i = 0$ on $U_i \cap U_j$ $\Rightarrow = 0$ in $S(\mathbb{P}^n) = K[x_0,\dots,x_n]$

$\Rightarrow \frac{g_i}{f_i} \overset{(*)}{=} \frac{g_j}{f_j} \in \mathrm{Frac}(K[x_0,\dots,x_n])$

$K[x_0,\dots,x_n]$ is UFD $\rightsquigarrow$ can assume f_i, g_i coprime, then element $\frac{g}{f} \in \mathrm{Frac}(K[x_0,\dots,x_n])$ in $(*)$ unique, $\varphi = \frac{g}{f}$ on all U_i

$\Rightarrow \varphi = \Phi\left(\frac{g}{f}\right)$.

(d) $e \in \mathbb{Z} \rightsquigarrow$ have multiplication maps

$$\mathcal{O}_{\mathbb{P}^n}(d)(U) \times \mathcal{O}_{\mathbb{P}^n}(e)(U) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d+e)(U), \ (\varphi, \psi) \mapsto \varphi \cdot \psi$$

$e = 0 \rightsquigarrow \mathcal{O}_{\mathbb{P}^n}(d)$ is sheaf of modules on $\mathbb{P}^n$

twisting sheaf on $\mathbb{P}^n$

Exa (a) Global sections:

$g/f \in \mathcal{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n) \rightsquigarrow V(f)=\emptyset \xRightarrow{\text{Nullstellens.}} f\in K^\times \rightsquigarrow g/f \in K[x_0,\dots,x_n]_d$

$$\Rightarrow \mathcal{O}_{\mathbb{P}^n}(d)(\mathbb{P}^n) = \begin{cases} K[x_0,\dots,x_n]_d & , d\geq 0 \\ \{0\} & , d<0 \end{cases}$$

(b) $U_0 = \{(x_0:x_1)\in\mathbb{P}^1 : x_0\neq 0\} \rightsquigarrow \frac{1}{x_0} \in \mathcal{O}_{\mathbb{P}^1}(-1)(U_0)$

(c) $f\in K[x_0,\dots,x_n]_e \rightsquigarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d+e)$ morph. of sheaves

$\varphi \longmapsto \varphi\cdot f$

(d) $d\neq 0 \rightsquigarrow \mathcal{O}_{\mathbb{P}^n}(d)$ and $\mathcal{O}_{\mathbb{P}^n}$ not isomorphic (global sect. differ!)

<u>But</u> for $U_i = \{x : x_i\neq 0\} \subseteq \mathbb{P}^n$ we have isom.

$$\mathcal{O}_{\mathbb{P}^n}|_{U_i} \xrightarrow[\sim]{f} \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \quad \text{w/ inverse} \quad \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \xrightarrow[\sim]{f^{-1}} \mathcal{O}_{\mathbb{P}^n}|_{U_i}$$

$$\varphi \longmapsto \varphi\cdot x_i^d \qquad\qquad \varphi \longmapsto \varphi\cdot x_i^{-d}$$

<u>Exercise</u> (tautological sheaf of $\mathbb{P}^n$)

$$\mathcal{F}(U) = \{\varphi : U \to K^{n+1} \text{ morphism} : \varphi(L)\in L \ \forall L\in U\}$$

$U\subseteq\mathbb{P}^n$ open

closed pt.

<u>tautological sheaf</u>

<u>Prove</u> $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(-1)$.

Push-forward of sheaves

Goal of rest of chapter

show common constructions with sheaves (of modules)

Helpful tool:

Construction (Morphisms on stalks)

$\mathcal{F}$ sheaf of modules on X $\rightsquigarrow$ $\mathcal{F}_p$ is $\mathcal{O}_{X,p}$-module $\forall p \in X$

$\rightsquigarrow$ $\mathcal{F} \xrightarrow{f} \mathcal{G}$ morph. of $\mathcal{O}_X$-modules $\Rightarrow$ $\mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p$ is $\mathcal{O}_{X,p}$-mod. hom.

$[(U,\varphi)] \mapsto [(U, f_U(\varphi))]$

Exercise $\mathcal{F} \xrightarrow{f} \mathcal{G}$ isomorphism $\Leftrightarrow$ $\mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p$ isomorphism $\forall p \in X$.

To construct new sheaves:

→ stay on one space X, work with sheaves $\mathcal{F}, \mathcal{G}$, $\mathcal{F} \to \mathcal{G}$

↳ sum, tensor product, kernel, cokernel, ... (later)

→ consider morphism $X \xrightarrow{f} Y$ and sheaves on X, Y

↳ push-forward, pull-back

Def (Push-forward of sheaves)

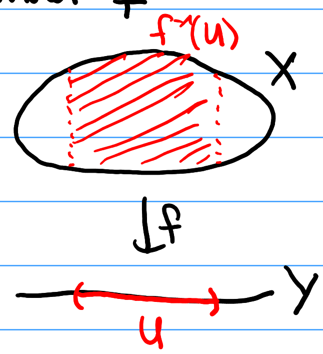
$f: X \to Y$ morphism of schemes, $\mathcal{F}$ sheaf on X

$\rightsquigarrow$ $(f_* \mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$ for $U \subseteq Y$ open

defines sheaf on Y: push-forward of $\mathcal{F}$ under f

In fact: $f_* \mathcal{F}$ is $\mathcal{O}_Y$-module

$\lambda \cdot \varphi = (f_U^* \lambda) \cdot \varphi$ for $\lambda \in \mathcal{O}_Y(U)$, $\varphi \in \mathcal{F}(f^{-1}(U))$



Exa

(a) $f: X \to Y$ morphism of schemes

$\rightsquigarrow f_U^* : \mathcal{O}_Y(U) \to \underbrace{\mathcal{O}_X(f^{-1}(U))}_{=(f_* \mathcal{O}_X)(U)} \quad \rightsquigarrow f^* : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$

morphism of sheaves

Def' A morphism between loc. ringed spaces X, Y is data of

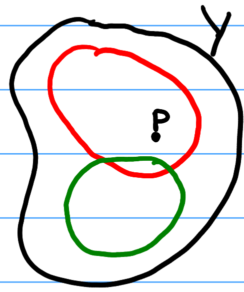
$$(\underbrace{f: X \longrightarrow Y}_{\text{continuous map}}, \underbrace{f^* : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X}_{\text{morphism of sheaves of rings on } Y})$$

such that $(f_p^*)^{-1}(I_p) = I_{f(p)}$ with $f_p^* : \mathcal{O}_{Y, f(p)} \longrightarrow (f_* \mathcal{O}_X)_{f(p)} \rightsquigarrow \mathcal{O}_{X,p}$

(b) Y variety, $p \in Y$ (closed) point, $i : \{p\} \to Y$ inclusion

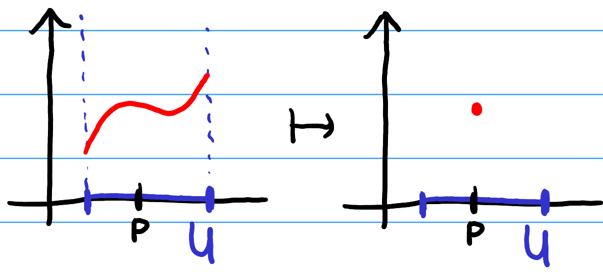
$\rightsquigarrow \mathcal{O}_{\{p\}}(\{p\}) = K, \; \mathcal{O}_{\{p\}}(\emptyset) = \{0\}$

$\Rightarrow (i_* \mathcal{O}_{\{p\}})(U) = \mathcal{O}_{\{p\}}(i^{-1}(U)) = \begin{cases} K, & p \in U \\ \{0\}, & p \notin U \end{cases}$



$i^* : \mathcal{O}_Y(U) \longrightarrow i_* \mathcal{O}_{\{p\}}(U)$

$\varphi \longmapsto \begin{cases} \varphi(p), & p \in U \\ 0, & p \notin U \end{cases}$



$i_* \mathcal{O}_{\{p\}} =: K_p$ skyscraper sheaf on Y at p

Kernel sheaves and image presheaves

Commutative algebra

R ring, M, N R-modules, $f: M \to N$ R-module hom.

$\rightsquigarrow \operatorname{Ker}(f) \subseteq M$ and $\operatorname{im}(f) \subseteq N$ R-submodules

<u>Q</u> How to generalize to morphism $f: \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules?

<u>Construction</u> (Kernel sheaf)

$f: \mathcal{F} \to \mathcal{G}$ morphism of sheaves on scheme X

$(\operatorname{Ker} f)(U) := \operatorname{Ker}(f_U: \mathcal{F}(U) \to \mathcal{G}(U))$ for $U \subseteq X$ open

$\rightsquigarrow$ presheaf (with restriction from $\mathcal{F}$)

$\varphi \in \operatorname{Ker}(f_U)$, $V \subseteq U \rightsquigarrow f_V(\varphi|_V) = f_U(\varphi)|_V = 0|_V = 0$

$\rightsquigarrow \varphi|_V \in \operatorname{Ker}(f_V)$

$\operatorname{Ker}(f)$ is a sheaf:

$\{U_i : i \in I\}$ open cover of U, $\varphi_i \in \operatorname{Ker}(f_{U_i}) \subseteq \mathcal{F}(U_i)$
compatible on overlaps $U_i \cap U_j$

$\xRightarrow{\mathcal{F}\text{ sheaf}} \exists!\, \varphi \in \mathcal{F}(U) : \varphi|_{U_i} = \varphi_i$

Claim: $\varphi \in \operatorname{Ker}(f_U)$: $\psi = f_U(\varphi)$ satisfies $\psi|_{U_i} = f_{U_i}(\varphi|_{U_i}) = 0$

$\xRightarrow{\mathcal{G}\text{ sheaf}} \psi = 0 \in \mathcal{G}(U) \Rightarrow \varphi \in \operatorname{Ker}(f_U)$.

$\operatorname{Ker}(f)$: <u>Kernel sheaf</u>

<u>Idea</u> Being in $\operatorname{Ker}(f_U)$ can be checked locally on U

Construction (Image presheaf)

$f: \mathcal{F} \to \mathcal{G}$ morphism of sheaves on scheme X

$$(\mathrm{Im}' f)(U) := \mathrm{Im}\,(f_U : \mathcal{F}(U) \to \mathcal{G}(U)) \quad \text{for } U \subseteq X \text{ open}$$

$\rightsquigarrow$ <u>image presheaf</u> (with restriction from $\mathcal{G}$)

$\psi = f_U(\varphi) \in \mathrm{Im}(f_U)$, $V \subseteq U \rightsquigarrow f_V(\varphi|_V) = f_U(\varphi)|_V = \psi|_V$

$\rightsquigarrow \psi|_V \in \mathrm{Im}(f_V)$

Problem Not a sheaf in general

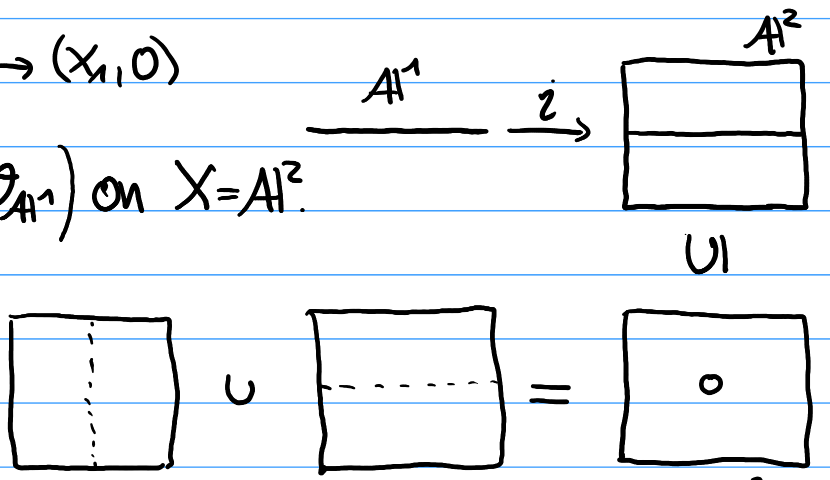
<u>Idea</u> $\{U_i : i \in I\}$ cover of U, $\psi_i = f_{U_i}(\varphi_i) \in (\mathrm{Im}' f)(U_i)$

$\rightsquigarrow \varphi_i \in \mathcal{F}(U_i)$ don't have to glue together to $\varphi \in \mathcal{F}(U)$!

only know ψ_i agree on overlaps, not necess. the φ_i !

<u>Exa</u> $i: \mathbb{A}^1 \to \mathbb{A}^2$, $x_1 \to (x_1, 0)$

$\rightsquigarrow f = (i^*: \mathcal{O}_{\mathbb{A}^2} \to i_* \mathcal{O}_{\mathbb{A}^1})$ on $X = \mathbb{A}^2$.



$U_1 = D(x_1)$ $\qquad$ $U_2 = D(x_2)$ $\qquad$ $U = \mathbb{A}^2 \setminus \{0\}$

$V \subseteq U$ open	$\mathcal{O}_{\mathbb{A}^2}(V)$	$(i_* \mathcal{O}_{\mathbb{A}^1})(V)$	$(\mathrm{Im}' f)(V)$
U_1	$K[x_1,x_2]_{x_1}$	$K[x_1]_{x_1}$	$K[x_1]_{x_1}$
U_2	$K[x_1,x_2]_{x_2}$	$\{0\}$	$\{0\}$
$U_1 \cap U_2$	$K[x_1,x_2]_{x_1 x_2}$	$\{0\}$	$\{0\}$
$U = U_1 \cup U_2$	$K[x_1,x_2]$	$K[x_1]_{x_1}$	$K[x_1]$

$\rightsquigarrow \frac{1}{x_1} \in (\mathrm{Im}' f)(U_1)$, $0 \in (\mathrm{Im}' f)(U_2)$ compatible, but don't glue on U.

The tensor presheaf

Saw $\mathcal{F} \xrightarrow{f} \mathcal{G}$ map of $\mathcal{O}_X$-modules

$\rightsquigarrow$ $\ker(f)$ sheaf of modules, $\operatorname{Im}'(f)$ only presheaf of modules

Construction (Tensor presheaf)

$\mathcal{F}, \mathcal{G}$ $\mathcal{O}_X$-modules $\rightsquigarrow$ define <u>tensor presheaf</u> $\mathcal{F} \otimes' \mathcal{G}$ by

$$(\mathcal{F} \otimes' \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \quad \text{for } U \subseteq X \text{ open.}$$

Problem $\mathcal{F} \otimes' \mathcal{G}$ not a sheaf in general:

$X = \mathbb{P}^1$, $\mathcal{F} = \mathcal{O}_X(1)$, $\mathcal{G} = \mathcal{O}_X(-1)$

$U_i = \{(x_0 : x_1) \in \mathbb{P}^1 : x_i \neq 0\}$ for $i = 0, 1$

$\rightsquigarrow$ $x_0 \otimes \frac{1}{x_0} \in (\mathcal{F} \otimes' \mathcal{G})(U_0)$ and $x_1 \otimes \frac{1}{x_1} \in (\mathcal{F} \otimes' \mathcal{G})(U_1)$

Compatible on $U_0 \cap U_1 = D(x_0 x_1)$:

$$x_0 \otimes \frac{1}{x_0} = \underbrace{\frac{x_1}{x_0} \cdot \frac{x_0}{x_1}}_{\text{regular on } U_0 \cap U_1} \cdot x_0 \otimes \frac{1}{x_0} = \left(\frac{x_1}{x_0} \cdot x_0\right) \otimes \left(\frac{x_0}{x_1} \cdot \frac{1}{x_0}\right) = x_1 \otimes \frac{1}{x_1} \in (\mathcal{F} \otimes' \mathcal{G})(U_0 \cap U_1)$$

If $\mathcal{F} \otimes' \mathcal{G}$ was a sheaf $\rightsquigarrow$ would glue to unique section in

$$(\mathcal{F} \otimes' \mathcal{G})(X) = \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \underbrace{\mathcal{G}(X)}_{=\{0\}} = \{0\}$$

$\lightning$ Since $0 \neq x_0 \otimes \frac{1}{x_0} \in (\mathcal{F} \otimes' \mathcal{G})(U_0) = \underbrace{(K[x_0] \cdot x_0)}_{=\mathcal{F}(U_0)} \otimes_{\underbrace{K[x_0]}_{=\mathcal{O}_X(U_0)}} \underbrace{(K[x_0] \cdot \tfrac{1}{x_0})}_{=\mathcal{G}(U_0)} = K[x_0] \cdot x_0 \otimes \frac{1}{x_0}$

$$\boxed{\begin{array}{ll} \mathcal{O}_{\mathbb{P}^n}|_{U_i} \xrightarrow[\sim]{f} \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \text{ w/ inverse } & \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \xrightarrow[\sim]{f^{-1}} \mathcal{O}_{\mathbb{P}^n}|_{U_i} \\ \varphi \longmapsto \varphi \cdot x_i^d & \varphi \longmapsto \varphi \cdot x_i^{-d} \end{array}}$$

$\rightsquigarrow$ need way to convert presheaf into a sheaf!

Sheafification

$\mathcal{F}'$ presheaf $\rightsquigarrow$ get sheaf $\mathcal{F} = (\mathcal{F}')^{sh}$ by taking stalks of $\mathcal{F}'$ and requiring them to fit together nicely

stalks of the presheaf $\mathcal{F}'$

the sheafification $\mathcal{F} = (\mathcal{F}')^{sh}$

Def (Sheafification)

$\mathcal{F}'$ presheaf on X. For $U \subseteq X$ open, set

$$\mathcal{F}(U) := \left\{ \varphi = (\varphi_p)_{p\in U} : \begin{array}{l} \varphi_p \in \mathcal{F}'_p \text{ for all } p \in U, \\ \text{and } \forall p \in U \ \exists \text{ open nbhd. } U_p \subseteq U \\ \text{and section } s \in \mathcal{F}'(U_p) \\ \text{with} \\ \quad \varphi_q = s_q \in \mathcal{F}'_q \quad \forall q \in U_p \end{array} \right\}$$

$s_q = [(U_p, s)]$ germ of s at q

Condition on φ_p local on U

$\rightsquigarrow$ $\mathcal{F} =: (\mathcal{F}')^{sh}$ is a sheaf on X

$\searrow$ sheafification of $\mathcal{F}'$.

Exa (a) X affine variety $\rightsquigarrow$ get presheaf $\mathcal{O}'_X$ by global quot. of polynomials:

$$\mathcal{O}'_X(U) = \{ \varphi : U \to K : \exists f, g \in A(X) \text{ w/ } \varphi = \tfrac{f}{g} \text{ on } U \}$$

$g(x) \neq 0 \ \forall x \in U$

$\rightsquigarrow$ [Exa 3.3] $\mathcal{O}'_X$ is not a sheaf

But: $\mathcal{O}_X = (\mathcal{O}'_X)^{sh}$ is sheafification.

(b) $X = \mathbb{P}^1$, $s_i = x_i \otimes \frac{1}{x_i} \in (\mathcal{O}_X(1) \otimes' \mathcal{O}_X(-1))(U_i)$ as above

$\rightsquigarrow$ glue to global section $\varphi = (\varphi_p)_{p \in \mathbb{P}^1}$ of $(\mathcal{O}_X(1) \otimes' \mathcal{O}_X(-1))^{sh}$

$$\varphi_p = \begin{cases} (s_0)_p, & p \in U_0 \\ (s_1)_p, & p \in U_1 \end{cases}$$

Check before: $(s_0)_p = (s_1)_p$ for $p \in U_0 \cap U_1$

<u>Rmk</u> $\mathcal{F}'$ presheaf on X $\rightsquigarrow$ get morphism $\theta : \mathcal{F}' \to \mathcal{F} = (\mathcal{F}')^{sh}$ by

$\theta_U : \mathcal{F}'(U) \longrightarrow \mathcal{F}(U)$, $s \longmapsto (s_p)_{p \in U}$. for $U \subseteq X$ open

<u>Lem</u> (Properties of sheafification)
$\mathcal{F}'$ presheaf on X, $\mathcal{F} = (\mathcal{F}')^{sh}$ sheafification, $\theta : \mathcal{F}' \to \mathcal{F}$ as above.
(a) $\forall p \in X$: $\theta_p : \mathcal{F}'_p \xrightarrow{\sim} \mathcal{F}_p$ isom. of stalks
(b) $\mathcal{F}'$ sheaf $\Rightarrow$ θ is isomorphism $\mathcal{F}' \cong \mathcal{F}$

<u>Pf</u> (a) $\mathcal{F}_p \xrightarrow{\mathcal{S}_p} \mathcal{F}'_p$, $[(U, \varphi = (\varphi_q)_{q \in U})] \longmapsto \varphi_p$ map in other direction
$\mathcal{S}_p \circ \theta_p = \mathrm{id}$ clear, for other direction: $[(U, \varphi = (\varphi_q)_q)] = [(U', \varphi = (s_q))]$
shrink U $\quad s \in \mathcal{F}'(U')$

$\rightsquigarrow (\theta_p \circ \mathcal{S}_p)([(U', \varphi = (s_q)_q)]) = \theta_p(s_p) = [(U', (s_q)_q)]$.

(b) $\mathcal{F}'$ sheaf $\Rightarrow$ θ morphism of sheaves inducing isom θ_p $\forall p \in X$
$\xRightarrow{[Ex. 1.38]}$ θ isomorphism. $\square$

<u>Exercise</u> (Universal property of sheafification)
$\mathcal{F}'$ presheaf on X, $\mathcal{F}' \xrightarrow{\theta} \mathcal{F} = (\mathcal{F}')^{sh}$.
Then for any morphism $\mathcal{F}' \xrightarrow{f'} \mathcal{G}$ to a sheaf
we have a unique factorization

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "\theta"] \arrow[dr, "\forall f'"'] & \mathcal{F} \arrow[d, dashed, "\exists! f"] \\ & \mathcal{G} \end{tikzcd}$$

morphism of sheaves

Constructions with sheaves

Def $\mathcal{F}, \mathcal{G}$ sheaves of $\mathcal{O}_X$-modules on scheme X.

(a) $\mathcal{F} \xrightarrow{f} \mathcal{G}$ morphism of $\mathcal{O}_X$-modules

$\operatorname{Im} f = (\operatorname{Im}' f)^{sh}$ <u>image sheaf</u> $\xrightarrow{\text{univ. property}}$

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, "f"'] & \operatorname{Im}' f \arrow[r, "\theta"] \arrow[d] & \operatorname{Im} f \arrow[dl, dashed, "\exists!"] \\ & \mathcal{G} & \end{tikzcd}$$

f <u>injective</u> $:\Leftrightarrow \ker(f) = 0$ ← 0-sheaf

f <u>surjective</u> $:\Leftrightarrow \operatorname{Im}(f) \xrightarrow{\tilde{f}} \mathcal{G}$ is an isomorphism

(b) $\mathcal{F} \xrightarrow{f} \mathcal{G}$ injective $\rightsquigarrow \mathcal{F}(U) \leq \mathcal{G}(U)$ sub-$\mathcal{O}(U)$-module

get presheaf $(\mathcal{G}/'\mathcal{F})(U) = \mathcal{G}(U)/\mathcal{F}(U)$

$\rightsquigarrow$ sheaf $\mathcal{G}/\mathcal{F} = (\mathcal{G}/'\mathcal{F})^{sh}$ <u>quotient sheaf</u>

(c) A sequence

$$\cdots \longrightarrow \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \xrightarrow{f_{i+1}} \mathcal{F}_{i+2} \longrightarrow \cdots \qquad (*)$$

of morphisms of $\mathcal{O}_X$-modules is a <u>complex</u> if $f_{i+1} \circ f_i = 0 \ \forall i$.

↳ induces map $\operatorname{Im}(f_i) \longrightarrow \ker(f_{i+1})$ (univ. property)

Say sequence $(*)$ is <u>exact</u> if $\operatorname{Im}(f_i) \xrightarrow{\sim} \ker(f_{i+1})$ is isomorphism.

(d) <u>tensor sheaf</u> $\mathcal{F} \otimes \mathcal{G} := (\mathcal{F} \otimes' \mathcal{G})^{sh}$

(e) <u>dual sheaf</u> $\mathcal{F}^\vee$ = sheafification of $(U \mapsto \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{O}_U))$

hom. of $\mathcal{O}_U$-modules

Note cannot use $(U \mapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{O}_X(U)))$

$\rightsquigarrow$ not a presheaf!

Exact sequences of sheaves of modules

Have defined A complex of morphisms of $\mathcal{O}_X$-modules

$$\cdots \longrightarrow \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \xrightarrow{f_{i+1}} \mathcal{F}_{i+2} \longrightarrow \cdots \qquad (*)$$

is exact if $\operatorname{Im}(f_i) \longrightarrow \operatorname{Ker}(f_{i+1})$ is an isom. $\forall i$.

Exerc. $\mathcal{F} \xrightarrow{f} \mathcal{G}$ morph. of $\mathcal{O}_X$-modules on scheme X.
$\mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p$ induced morphism of stalks at $p \in X$.
Show:
(a) $(\operatorname{Ker} f)_p = \operatorname{Ker}(f_p)$ (b) $(\operatorname{Im} f)_p = \operatorname{Im}(f_p)$.

$\rightsquigarrow$ helps with checking exactness (e.g. f surjective $\not\Rightarrow$ $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ surjective $\forall U \subseteq X$ open)

Lem Consider a sequence of morphisms of $\mathcal{O}_X$-modules

$$\cdots \longrightarrow \mathcal{F}_i \xrightarrow{f_i} \mathcal{F}_{i+1} \xrightarrow{f_{i+1}} \mathcal{F}_{i+2} \longrightarrow \cdots \qquad (*)$$

on a scheme X. Then the following are equivalent:
(a) The sequence $(*)$ is exact.
(b) The restricted sequence $\dashrightarrow \mathcal{F}_i|_U \xrightarrow{f_i|_U} \mathcal{F}_{i+1}|_U \to \cdots$ is exact for all $U \subseteq X$ open.
(c) The restricted sequence $\dashrightarrow \mathcal{F}_i|_{U_k} \xrightarrow{f_i|_{U_k}} \mathcal{F}_{i+1}|_{U_k} \to \cdots$ is exact for some open cover $\{U_k : k \in I\}$ of X.
(d) The induced sequence $\dashrightarrow (\mathcal{F}_i)_p \xrightarrow{(f_i)_p} (\mathcal{F}_{i+1})_p \to \cdots$ is exact $\forall p \in X$.

Pf $\boxed{(a) \Rightarrow (d)}$ $(*)$ exact $\Rightarrow \operatorname{Im}(f_i) \xrightarrow{\sim} \operatorname{Ker}(f_{i+1})$

$$\Rightarrow \operatorname{Im}(f_i)_p \xrightarrow{\sim} \operatorname{Ker}(f_{i+1})_p \quad \forall p$$

$$\begin{array}{ccc} \| & & \| \\ \operatorname{Im}((f_i)_p) & & \operatorname{Ker}((f_{i+1})_p) \end{array}$$

Exercise above

$$\Rightarrow \dashrightarrow (\mathcal{F}_i)_p \xrightarrow{f_{i,p}} (\mathcal{F}_{i+1})_p \to \cdots \text{ is exact.} \qquad \#$$

$\boxed{(d)\Rightarrow(a)}$ First: show $(*)$ is complex

$\mathcal{F}_i \underset{0}{\xrightarrow{f_{i+1}\circ f_i}} \mathcal{F}_{i+2}$ two morph. of sheaves, $(f_{i+1}\circ f_i)_p = (f_{i+1})_p \circ (f_i)_p = 0$

$\rightsquigarrow$ agree on all stalks $\Longrightarrow f_{i+1}\circ f_i = 0$ see $\mathcal{F}_i = \mathcal{F}_i^{sh}$, $\mathcal{F}_{i+2} = \mathcal{F}_{i+2}^{sh}$

$(*)$ complex $\Rightarrow \underbrace{\mathrm{Im}(f_i) \longrightarrow \mathrm{Ker}(f_{i+1})}_{\text{isom at all stalks (Exercise again)}} \Rightarrow \mathrm{Im}(f_i) \cong \mathrm{Ker}(f_{i+1})$

#

$\boxed{(b)\Leftrightarrow(d) \text{ and } (c)\Leftrightarrow(d)}$ Same as $(a)\Leftrightarrow(d)$ since stalks at p can be calculated on any open $p\in U\subset X$

□

<u>Exa</u> (Skyscraper sequence)

(a) $p=(1:0)\in X=\mathbb{P}^1$, $i:\{p\}\longrightarrow X$ inclusion

$$\rightsquigarrow \quad 0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{f} \mathcal{O}_X \xrightarrow{g} K_p \longrightarrow 0 \qquad (\star)$$

f: multiplic. by x_1 [Exa 13.5(c)]

$g: \mathcal{O}_X \longrightarrow i_* \mathcal{O}_{\{p\}}$ evaluation at p [Exa. 13.10(b)]

<u>Claim</u> $(\star)$ is exact.

- $\mathrm{Ker}(f)\ni\varphi$ must satisfy $\varphi\cdot x_1 = 0 \in \mathrm{Frac}(K[x_0,x_1])$
  $\Rightarrow \varphi = 0 \Rightarrow \mathrm{Ker}(f) = 0$-sheaf

- $\mathrm{Im}'(f)$ = regular functions of form φx_1 = funct. vanish. at p
  $\hookrightarrow \mathrm{Im}'(f) = \mathrm{Im}(f)$ is sheaf (condition is local)
  $\Rightarrow \mathrm{Im}(f)(U) = \{\varphi\in\mathcal{O}_X(U) : \varphi(p)=0 \text{ if } p\in U\} = (\mathrm{Ker}\, g)(U)$

- $\mathrm{Im}'(g) = K_p$ (via const. functions) $\Rightarrow \mathrm{Im}'(g) = \mathrm{Im}(g) = K_p \Rightarrow g$ surjective

(b) $q = (0:1) \in \mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \xrightarrow{\cdot x_0 x_1} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{h} K_p \oplus K_q \longrightarrow 0$$

- still exact (check on $U_i = \mathbb{P}^1 \setminus V(x_i)$ + Lemma)
- h surjective but

$$h_{\mathbb{P}^1} : \underbrace{\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)}_{=K} \xrightarrow[a \mapsto (a,a)]{} \underbrace{(K_p \oplus K_q)(\mathbb{P}^1)}_{=K^2} \qquad \text{not surjective!}$$

<u>Exa</u> (Tensor products of twisting sheaves)

$$(\mathcal{O}_{\mathbb{P}^n}(d))(U) \otimes_{\mathcal{O}_{\mathbb{P}^n}(U)} (\mathcal{O}_{\mathbb{P}^n}(e))(U) \xrightarrow{m_U} (\mathcal{O}_{\mathbb{P}^n}(d+e))(U), \quad \varphi \otimes \psi \mapsto \varphi \cdot \psi$$

are $\mathcal{O}_{\mathbb{P}^n}(U)$-module-homomorphisms

$\rightsquigarrow$

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^n}(d) \otimes' \mathcal{O}_{\mathbb{P}^n}(e) \arrow[d] \arrow[dr, "m"] & \\
\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\mathbb{P}^n}(e) \arrow[r, dashed, "\exists \hat{m}"'] & \mathcal{O}_{\mathbb{P}^n}(d+e)
\end{tikzcd}$$

univ. prop. of sheafification

On $U_i = \mathbb{P}^n \setminus V(x_i)$: $\hat{m}$ restricts to isom. [Exa. 13.5(d)]

$$\mathcal{O}_{\mathbb{P}^n}|_{U_i} \otimes \mathcal{O}_{\mathbb{P}^n}|_{U_i} \longrightarrow \mathcal{O}_{\mathbb{P}^n}|_{U_i}$$

$\Rightarrow$ $\hat{m}$ isom. by Lemma.